Examples Show $\sqrt{1+\sqrt{2}}$ is algebraic over $\mathbb{Q}$.

We need to find a poly. $p(x) \in \mathbb{Q}[x]$ s.t. $p(\sqrt{1+\sqrt{2}}) = 0$.

[Fun fact: If $\exists\, p(x)$ in $\mathbb{Q}[x]$ s.t. $p(\alpha)=0$, Also $\exists\, \tilde{p}(x) \in \mathbb{Z}[x]$ s.t. $\tilde{p}(\alpha)=0$. Why? Mult by LCM of the denoms of all the fraction coefficients.]

Let $x = \sqrt{1+\sqrt{2}}$

$$x^2 = 1+\sqrt{2}$$

$$x^2 - 1 = \sqrt{2}$$

$$(x^2-1)^2 = 2$$

Let $p(x) = (x^2-1)^2 - 2$. Then $p(\sqrt{1-\sqrt{2}}) = 0$

$p(x) = x^4 - 2x^2 - 1$. $\therefore \sqrt{1-\sqrt{2}}$ is alg over $\mathbb{Q}$. ☒

$p(x) \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$.

Question: Is $p(x) = x^4 - 2x^2 - 1$ the poly. of min. degree in $\mathbb{Q}[x]$ s.t. $p(\sqrt{1-\sqrt{2}})=0$? Note: any such polynomial $q(x)$ must satisfy $(x - \sqrt{1-\sqrt{2}}) \mid q(x)$. If $q(x)$ has min degree, the $q(x), p(x) \in \langle x - \sqrt{1-\sqrt{2}} \rangle$ in E = extension field. in $\mathbb{Q}[x]$ $p(x) = A(x)\,q(x) + r(x)$, where $r(x) = 0$ or $\deg(r(x)) < \deg(q(x))$

But then $\underbrace{p(x) - A(x)\,q(x)}_{r(x)}$ has the property that $r(\sqrt{1-\sqrt{2}}) = 0$

$$\Rightarrow r(x) = 0 \Rightarrow q(x) \mid p(x).$$

$\therefore$ if a smaller degree polynomial $q(x)$ satisfies $q(\sqrt{1-\sqrt{2}})=0$, then $q(x)$ must be a factor of $p(x) = x^4 - 2x^2 - 1$.

Does $p(x) \nearrow x^4-2x^2-1$ factor in $\mathbb{Q}[x]$?

eg $(x-\ )(\text{cubic})$ or $(x^2 - \cdots)(x^2 \ \cdots)$

Note: $p(x+1) = (x+1)^4 - 2(x+1)^2 - 1$

$= x^4+4x^3+6x^2+4x+1-2(x^2+2x+1) - 1$

$= x^4+4x^3+4x^2+0x-2 \leftarrow$ fits Eisenstein criterion $\Rightarrow$ irred.

We have show x^4-2x^2-1 is a min. deg poly in $\mathbb{Q}[x]$ such that $\alpha = \sqrt{1+\sqrt{2}}$ is a root.

Notice also that this is a monic polynomial (leading coefficient $= 1$)

$\therefore$ x^4-2x^2-1 is the unique monic poly. of min degree that has $\underbrace{\sqrt{1+\sqrt{2}}}_{\alpha}$ as a root.

$p(x) =$ "min poly of $\alpha = \sqrt{1+\sqrt{2}}$ over $\mathbb{Q}$" = "irred poly of α over $\mathbb{Q}$" $= \mathrm{irr}(\sqrt{1+\sqrt{2}}, \mathbb{Q})$

$\boxed{\text{Example}}$ Show that $\alpha = \sqrt{9+4\sqrt{2}}$ is alg. over $\mathbb{Q}$.
Find the min degree of the polynomial $p(x)$ s.t. $p(\alpha) = 0$.

Let $x = \sqrt{9+4\sqrt{2}} = \alpha$

$x^2 = 9+4\sqrt{2}$

$x^2 - 9 = 4\sqrt{2}$

$(x^2-9)^2 = 32$

Let $p(x) = (x^2-9)^2 - 32 \Rightarrow p(\alpha) = 0$.

$p(x) = x^4 - 18x^2 + 49$.

Is this min degree. Can we factor?

Rational root test: roots are $\pm 1, \pm 7$

7 does not work.

-7 does not work

No linear factors.

$$\begin{array}{r} 2401 \\ -\ 882 \\ \hline 1519 \end{array} \qquad \begin{array}{r} 18 \\ 49 \\ \hline \end{array}$$

Rational root test: Given $p(x) \in \mathbb{Z}[x]$

$$p(x) = a_0 + a_1 x + \cdots + a_k x^k .$$

If $p(x)$ has a rational root $\frac{r}{s}$, then

$$\frac{r}{s} = \frac{\pm(\text{factor of } a_0)}{\pm(\text{factor of } a_k)} .$$